BOUNDING THE EXPONENT OF A VERBAL SUBGROUP

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ABSTRACT. We deal with the following conjecture. If w is a group word and G is a finite group in which any nilpotent subgroup generated by w-values has exponent dividing e, then the exponent of the verbal subgroup $w(G)$ is bounded in terms of e and w only. We show that this is true in the case where w is either the nth Engel word or the word $[x^n, y_1, y_2, \ldots, y_k]$ (Theorem A). Further, we show that for any positive integer e there exists a number $k = k(e)$ such that if w is a word and G is a finite group in which any nilpotent subgroup generated by products of k values of the word w has exponent dividing e, then the exponent of the verbal subgroup $w(G)$ is bounded in terms of e and w only (Theorem B).

1. INTRODUCTION

If w is a group word in variables $x_1, x_2, \ldots, x_m$ we think of it as a function defined on any given group G. The subgroup of G generated by the values of w is called the verbal subgroup of G corresponding to the word w. This will be denoted $w(G)$. The study of verbal subgroups of groups is a classical topic of group theory. It dates back to the theory of varieties of groups and the work of P. Hall.

A number of outstanding results about words in finite groups have been obtained in recent years. In this context we mention Shalev's theorem that for any nontrivial group word w, every element of every sufficiently large finite simple group is a product of at most three w-values [15], and the proof by Liebeck, O'Brien, Shalev and Tiep [9] of Ore's conjecture: Every element of a finite simple group is a commutator. Another significant result is that of Nikolov and Segal that if G is an m-generated finite group, then every element of G' is a product of m-boundedly many commutators [11].

It was shown in [16] that if w is a multilinear commutator and G a finite group in which any nilpotent subgroup generated by w-values has exponent dividing e, then the exponent of the verbal subgroup $w(G)$ is bounded in terms of e and w only.

Recall that a group has exponent e if $x^e = 1$ for all $x \in G$ and e is the least positive integer with that property. Multilinear commutators

(outer commutator words) are words which are obtained by nesting commutators, but using always different indeterminates. For example, the word $[[x_1, x_2], [x_3, x_4, x_5], x_6]$ is a multilinear commutator. On the other hand, many important words are not multilinear commutators. In particular, the nth Engel word, defined inductively by

$$[x,_1 y] = [x, y] \ \text{ and } \ [x,_n y] = [[x,_{n-1} y], y],$$

is not a multilinear commutator when $n \geq 2$.

In view of the aforementioned result the following conjecture seems plausible.

CONJECTURE. *Let w be any word and G a finite group in which any nilpotent subgroup generated by w-values has exponent dividing e. Then the exponent of the verbal subgroup $w(G)$ is bounded in terms of e and w only.*

Recall that a commutator word is a word w such that $w(G) = 1$ for all abelian groups G; this is equivalent to the requirement that for each variable x appearing in w the sum of all exponents in the occurrences of x in w is zero. It is easy to see that for non-commutator words the above conjecture holds.

The result in [16] and our first result provide additional evidence in favor of this conjecture.

THEOREM A. *Let w be the nth Engel word or the word $[x^n, y_1, y_2, \ldots, y_k]$. Assume that G is a finite group in which any nilpotent subgroup generated by w-values has exponent dividing e. Then the exponent of the verbal subgroup $w(G)$ is bounded in terms of e and w only.*

So far we have been unable to prove the conjecture for arbitrary words w. Yet, we obtained a result in the same direction that deals with arbitrary words. If w is a word in m variables, we define inductively w_j as follows: $w_1 = w$ and $w_j = w_{j-1} \cdot w$, where the variables appearing in w_{j-1} and w are disjoint, so that w_j is a word in mj variables. For instance, if $w = [x, y, y]$, then $w_2 = [x_1, y_1, y_1][x_2, y_2, y_2]$. We remark that for any positive integer j, any group G and any word w we have $w(G) = w_j(G)$.

THEOREM B. *Given a positive integer e, there exists a number $k = k(e)$ such that if w is a word and G is a finite group in which any nilpotent subgroup generated by w_k-values has exponent dividing e, then the exponent of the verbal subgroup $w(G)$ is bounded in terms of e and w only.*

We emphasize that the hypothesis of Theorem B does not imply that the word w has bounded width in G. The reader can consult [14] for questions related to the width of a word in a finite group. It is easy to see that since we make no assumptions on the number of generators of G, the word w can have arbitrarily large width.

We make no attempts to write down explicit estimates for the exponents of $w(G)$ in our results. The next section is devoted to the proof of Theorem A. The proof of Theorem B will be given in Section 3. Both proofs depend

on the classification of finite simple groups and on Zelmanov's solution of the restricted Burnside problem [19, 20].

2. Theorem A

Throughout the paper we use the expression "$\{a,b, \ldots\}$-bounded" to mean "bounded from above by some function depending only on a,b $\ldots$". We say that a word w has the property (bb) if there exists a constant b depending on w such that every finite group H satisfying the law $w \equiv 1$ is an extension of a group of exponent dividing b by a group which is nilpotent of class at most b. We note that the Engel word has the property (bb) by the main theorem in [2]; the fact that the word $w = [x^n, y_1, y_2, \ldots, y_k]$ has the property (bb) will be proved shortly. As usual, $Z_i(H)$ and $\gamma_i(H)$ denote the ith term of the upper and lower central series of a group H, respectively. We use $\gamma_\infty(H)$ to denote the nilpotent residual of H, that is $\gamma_\infty(H) = \cap_{i \geq 1} \gamma_i(H)$.

The next lemma is Lemma 2.2 of [3]. In the case where $k = 1$ this is a well-known result, due to Mann [10].

LEMMA 2.1. *If G is a finite group such that $G/Z_k(G)$ has exponent m, then $\gamma_{k+1}(G)$ has $\{k,m\}$-bounded exponent.*

The fact that the word $w = [x^n, y_1, y_2, \ldots, y_k]$ has the property (bb) is now straightforward.

LEMMA 2.2. *The word $w = [x^n, y_1, y_2, \ldots, y_k]$ has the property (bb). In particular, if H is a finite group satisfying the law $w \equiv 1$, then $\gamma_{k+1}(H)$ has $\{k,n\}$-bounded exponent.*

Proof. We have $H^n \leq Z_k(H)$ and therefore $H/Z_k(H)$ has exponent dividing n. Thus, by Lemma 2.1, $\gamma_{k+1}(H)$ has $\{k,n\}$-bounded exponent. $\square$

If H is a subgroup of G, we write $w_{j,G}(H)$ for the subgroup generated by all w_j-values of G lying in H. We write $w_G(H)$ for $w_{1,G}(H)$.

The class of all finite groups in which every nilpotent subgroup generated by w_k-values has exponent dividing e will be denoted by $X(k,e,w)$. We write $X(e,w)$ in place of $X(1,e,w)$. If our conjecture holds, the exponents of all groups in $X(e,w)$ have a common bound. Our Theorem A says that this is so when w is either the nth Engel word or the word $[x^n, y_1, y_2, \ldots, y_k]$. Theorem B says that for any w and e there exists k such that the exponents of all groups in $X(k,e,w)$ have a common bound.

It seems that the class $X(e,w)$ is not closed with respect to quotients. Thus, it will be convenient to replace the condition defining the class $X(e,w)$ with a more manageable condition. In what follows $Y(e,w)$ will denote the class of finite groups G such that every w-value has order dividing e and $w(P)$ has exponent dividing e for every p-Sylow subgroup of G. It is clear that the class $Y(e,w)$ is closed under taking quotients of its members.

The next lemma is a straightforward consequence of [12, 5.2.5].

LEMMA 2.3. *Let w be a word with the property (bb) and G a finite group satisfying the law $w \equiv 1$. Assume that G is generated by elements of orders dividing r. Then G has (r, w)-bounded exponent.*

It is clear from the definitions that $X(e, w) \subseteq Y(e, w)$. In the case where w has the property (bb) an "almost converse" also holds.

LEMMA 2.4. *Assume that $e \geq 1$ and the word w has the property (bb). Then there exists an (e, w)-bounded integer f such that $Y(e, w) \subseteq X(f, w)$.*

Proof. Choose $G \in Y(e, w)$. Let H be a nilpotent subgroup of G generated by w-values. Since $H/w(H)$ satisfies the hypotheses of Lemma 2.3, it follows that $H/w(H)$ has (e, w)-bounded exponent. Moreover, if $H = P_1 \times \cdots \times P_r$, where the P_i's are the Sylow subgroups of H, we have $w(H) = w(P_1) \times \cdots \times w(P_r)$. As $G \in Y(e, w)$ the subgroup $w(P_i)$ has exponent dividing e for every i. It follows that H has (e, w)-bounded exponent, as required. $\square$

A tower of height r in a finite group G is a subgroup T of the form $T = P_1 \cdots P_r$ where

    (1) P_i is a p_i-group for $i = 1, \ldots, r$, with p_i a prime number.
    (2) P_i normalizes P_j for $i < j$.
    (3) $[P_i, P_{i-1}] = P_i$ for $i = 2, \ldots, r$.

Let us denote by $\mathrm{Fit}(G)$ the Fitting subgroup of G and by $F_i(G)$ the ith term of the upper Fitting series of G, defined recursively by $F_1(G) = \mathrm{Fit}(G)$ and $F_i(G)/F_{i-1}(G) = \mathrm{Fit}(G/F_{i-1}(G))$. If G is a finite soluble group, the least number h with the property that $F_h(G) = G$, is called the Fitting height of G. It is easy to see that the Fitting height of a tower $T = P_1 \cdots P_r$ equals precisely r (note that $P_2 \cdots P_r \leq \gamma_\infty(T)$). Moreover a finite soluble group G has Fitting height h if and only if h is the maximal number such that G possesses a tower of height h (see for instance Section 1 of [18]). A proof of the next lemma can be found in [16, Lemma 2.2].

LEMMA 2.5. *Let A be a group of automorphisms of a finite group G such that $(|A|, |G|) = 1$. Suppose that B is a normal subset of A such that $A = \langle B \rangle$ and let $k \geq 1$ be an integer. Then $[G, A]$ is generated by the subgroups of the form $[G, b_1, \ldots, b_k]$, where $b_1, \ldots, b_k \in B$.*

LEMMA 2.6. *Let w be the nth Engel word and $G \in Y(e, w)$. If $T = P_1 \cdots P_r$ is a tower of height r in G, then every p-Sylow subgroup of $P_2 \cdots P_r$ has (e, w)-bounded exponent.*

Proof. We will first show that for each $i = 2, \ldots, r$ the subgroup P_i is generated by w-values. Let $N = w_G(P_i)$. We wish to show that $N = P_i$. We put $\overline{P_{i-1}P_i} = P_{i-1}P_i/N$ and use the bar notation in the quotient. If $\bar{a} \in \bar{P}_{i-1}$ then $[\bar{P}_{i,n}\, \bar{a}] = 1$. Let $\bar{A} = \langle \bar{a} \rangle$. By Lemma 2.5 $[\bar{P}_i, \bar{A}]$ is generated by subgroups of the form $[\bar{P}_{i,n}\, \bar{a}]$, whence $[\bar{P}_i, \bar{A}] = 1$. This happens for every $\bar{a} \in \bar{P}_{i-1}$ and so $[\bar{P}_i, \bar{P}_{i-1}] = 1$. As $P_i = [P_i, P_{i-1}]$, it follows that $\bar{P}_i = 1$ and $P_i = N$, as required.

Now let P be a p-Sylow subgroup of $P_2 \cdots P_r$. We know that P is generated by w-values, which are all of orders dividing e. Since $G \in Y(e, w)$, it follows that the exponent of $w(P)$ divides e. Therefore the exponent of $w(P)$ is (e, w)-bounded. An application of Lemma 2.3 now completes the proof. $\qquad \square$

To prove an analogue of Lemma 2.6 for the word $w = [x^n, y_1, y_2, \ldots, y_k]$ we need a further technical lemma.

LEMMA 2.7. *Assume that $G = PQ$, where P is a p-group and Q is a q-group for different primes p and q. Assume further that P is normal in G and $G \in Y(e, w)$, where w is the word $w = [x^n, y_1, y_2, \ldots, y_k]$. Then the exponent of $\gamma_{k+1}(G)$ is (e, w)-bounded.*

Proof. Let $M = w_G(P)$. By Lemma 2.4 the group G and all of its quotients belong to $X(f, w)$ for some (e, w)-bounded integer f. As M is nilpotent, its exponent divides f. It is clear that $[P, _k G^n] \leq M$ and so the exponent of $[P, _k G^n]$ divides f, too. Passing to the quotient $G/[P, _k G^n]$ we can assume that $[P, _k G^n] = 1$ and so $G^n \cap P \leq Z_k(G^n)$. The group $G^n/G^n \cap P$ is a q-group and so it is nilpotent. Therefore G^n is nilpotent and we deduce that $[G^n, _k G]$ is of bounded exponent because it is a nilpotent subgroup generated by w-values and $G \in X(f, w)$. We can now pass to the quotient $G/[G^n, _k G]$ and without loss of generality assume that $w(G) = 1$. In view of Lemma 2.2 the result follows. $\qquad \square$

LEMMA 2.8. *Let $w = [x^n, y_1, y_2, \ldots, y_k]$ and let $G \in Y(e, w)$. If $T = P_1 \cdots P_r$ is a tower of height r in G, then every p-Sylow subgroup of $P_2 \cdots P_r$ has (e, w)-bounded exponent.*

Proof. We will first show that P_i is generated by elements of bounded orders for each $i = 2, \ldots, r$. As $P_i = [P_i, P_{i-1}]$ it follows from Lemma 2.5 that P_i is generated by subgroups of the form $[P_i, b_1, \ldots, b_k]$ with $b_1, \ldots, b_k \in P_{i-1}$. As the group $P_i P_{i-1}$ satisfies the hypotheses of Lemma 2.7, it follows that $\gamma_{k+1}(P_{i-1} P_i)$ has (e, w)-bounded exponent and this proves that P_i is generated by elements of bounded orders for each $i = 2, \ldots, r$. Now let P be a p-Sylow subgroup of $P_2 \cdots P_r$. Since $G \in Y(e, w)$, it follows that the exponent of $w(P)$ divides e. Since P is generated by elements of bounded orders, the result follows from Lemma 2.3. $\qquad \square$

LEMMA 2.9. *Let w be either the nth Engel word or the word $[x^n, y_1, y_2, \ldots, y_k]$. The Fitting height of any soluble group in $Y(e, w)$ is (e, w)-bounded.*

Proof. Let G be a soluble group in $Y(e, w)$ and choose a tower $T = P_1 \cdots P_r$ in G of height precisely the Fitting height of G. Of course, it is enough to prove that the Fitting height of the subgroup $P_2 \cdots P_r$ is (e, w)-bounded. By Lemmas 2.6 and 2.8 every p-Sylow subgroup of $P_2 \cdots P_r$ has (e, w)-bounded exponent. Therefore $P_2 \cdots P_r$ is a soluble group of (e, w)-bounded exponent. According to the Hall-Higman theory [7] the Fitting height of $P_2 \cdots P_r$ is bounded in terms of the exponent. This completes the proof. $\qquad \square$

We need another technical result concerning the nth Engel word.

LEMMA 2.10. *Let w be the nth Engel word and let T be a metanilpotent group. If M is the subgroup generated by all w-values contained in the Fitting subgroup of T, then T/M is nilpotent.*

Proof. Let F be the Fitting subgroup of T and let $g, a \in T$. As T/F is nilpotent, $[g,_r a] \in F$ for some positive integer r. Then $[g,_{r+n} a] \in M$. Therefore every element of T/M is left Engel. It follows that T/M is nilpotent (see for instance Proposition 12.3.3 of [12]), as required. □

PROPOSITION 2.11. *Let w be the nth Engel word or the word $[x^n, y_1, y_2, \ldots, y_k]$. Assume that G is a soluble group in $Y(e, w)$. Then the exponent of $w(G)$ is (e, w)-bounded.*

Proof. The proof is by induction on the Fitting height h of G, which is (e, w)-bounded by Lemma 2.9. If $h = 1$, then G is nilpotent and so the exponent of $w(G)$ divides e.

Now assume that $h > 1$ and let F be the Fitting subgroup of G. The subgroup $M = w_G(F)$ is nilpotent, so it has (e, w)-bounded exponent by Lemma 2.4.

Consider the case in which w is the nth Engel word. Let T/F be the Fitting subgroup of G/F. Then T is metanilpotent and by Lemma 2.10 T/M is nilpotent. Therefore the Fitting height of G/M is smaller than that of G. By induction $w(G/M)$ has (e, w)-bounded exponent. Hence, the exponent of $w(G)$ is (e, w)-bounded as well.

Now let $w = [x^n, y_1, y_2, \ldots, y_k]$. Put $K = G^n$. In the quotient $\bar{G} = G/M$ we have $[\bar{K}, \bar{F},_{k-1} \bar{G}] = 1$, whence $\bar{F} \cap \bar{K} \le Z_k(\bar{K})$. This implies that $\bar{K}$ has smaller Fitting height than G. By induction $w(\bar{K})$ has (e, w)-bounded exponent. Since the exponent of M is bounded as well, we deduce that the exponent of $w(K)M$ is bounded. Passing to the quotient over $w(K)M$, we may assume that $w(K) = 1$. It follows that K^n is nilpotent. By Lemma 2.4 $G \in X(f, w)$ for some (e, w)-bounded integer f. Therefore the subgroup $w_G(K^n)$ has exponent dividing f. Passing to the quotient over $w_G(K^n)$, we can assume that $[K^n,_k G] = 1$. This implies that $K^n \le Z_k(G)$, that is $G^{n^2} \le Z_k(G)$. Thus $G/Z_k(G)$ has bounded exponent. By Lemma 2.1 $\gamma_{k+1}(G)$ has bounded exponent, and as $w(G) \le \gamma_{k+1}(G)$ the result follows. □

Following the terminology used by Hall and Highman in [7], we say that a group G is monolithic if it has a unique minimal normal subgroup which is a non-abelian simple group. Nowadays very often such groups are called almost simple.

LEMMA 2.12. *Let L be a residually monolithic group satisfying a law $u \equiv 1$. Then L has u-bounded exponent.*

Proof. Observe that every monolithic group is isomorphic to a subgroup of $\mathrm{Aut}(S)$, where S is the non-abelian finite simple group isomorphic to the unique minimal normal subgroup of the group. A result of Jones [8] says

that any infinite family of finite simple groups generates the variety of all groups, so there are only finitely many finite simple groups satisfying the law $u \equiv 1$. This implies the existence of a bound (depending only on u) for the exponent of $\mathrm{Aut}(S)$, where S ranges through such simple groups. Thus, L has bounded exponent. $\qquad\square$

Let G be a finite group and r a positive integer. As in [17] we will associate with G a triple of numerical parameters $n_r(G) = (\lambda, \nu, \mu)$ where the parameters λ, μ, ν are defined as follows.

We recall that the nth derived word δ_n is defined recursively by: $\delta_1(x_1, x_2) = [x_1, x_2]$ and $\delta_i(x_1, \ldots, x_{2^i}) = [\delta_{i-1}(x_1, \ldots, x_{2^{i-1}}), \delta_{i-1}(x_{2^{i-1}+1}, \ldots, x_{2^i})]$ for all $i > 1$. Let $X_n(G)$ be the set of δ_n-values. If G is of odd order, we set $\lambda = \mu = \nu = 0$. Suppose that G is of even order and choose a 2-Sylow subgroup P in G. If the derived length $\mathrm{dl}(P)$ of P is at most $r+1$ we define $\lambda = \mathrm{dl}(P) - 1$. Put $\mu = 2$ if $X_\lambda(P)$ contains elements of order greater than two and $\mu = 1$ otherwise. We let $\nu = \mu$ if $X_\lambda(P) \nsubseteq Z(P)$ and $\nu = 0$ if $X_\lambda(P) \subseteq Z(P)$.

If the derived length of P is at least $r+2$ we define $\lambda = r$. Then μ will denote the number with the property that 2^μ is the maximum of orders of elements in $X_r(P)$. Finally, let 2^ν be the maximum of orders of commutators $[a, b]$, where $b \in P$ and $a = c^{\mu-1}$ for some element c of maximal order in $X_r(P)$.

We remark that our definition of μ is slightly different from that given in [17]. We believe that the definition in [17] was somewhat inaccurate while the present definition corrects the error. This minor correction does not require any changes in the subsequent arguments. In particular, the proof of Proposition 2.14 below remains unaffected.

The set of all possible triples $n_r(G)$ is naturally endowed with the lexicographic order. Moreover, if N is a normal subgroup of G, then $n_r(G/N) \leq n_r(G)$. As we will use induction on $n_r(G)$, we need to show that for some suitable r, depending only on e and w, there are only finitely many triples $n_r(G)$ associated with groups G in $Y(e, w)$.

LEMMA 2.13. *Let w be a word with the property (bb) and let $e \geq 1$. There exist (e, w)-bounded numbers $r, \lambda_0, \mu_0, \nu_0$ such that $n_r(G) \leq (\lambda_0, \mu_0, \nu_0)$ for every group $G \in Y(e, w)$.*

Proof. Choose $G \in Y(e, w)$. Let P be a 2-Sylow subgroup of G. We will show that for a suitable (e, w)-bounded r the exponent of $P^{(r)}$ is (e, w)-bounded, where $P^{(r)}$ is the rth term of the derived series of P. By the hypothesis $w(P)$ has exponent dividing e. Since w has the property (bb), it follows that $P/w(P)$ is an extension of a group of bounded exponent by a group of bounded nilpotency class. Therefore there exists an (e, w)-bounded number r such that $P^{(r)}$ is of bounded exponent. If $n_r(G) = (\lambda, \mu, \nu)$, the definitions imply that $\lambda \leq r$ and μ, ν are bounded by the exponent of $P^{(r)}$. $\qquad\square$

For the proof of the next proposition see [17].

PROPOSITION 2.14. *Let $r \geq 1$ and let G be a group of even order such that G has no nontrivial normal soluble subgroups. Then G possesses a normal subgroup L such that L is residually monolithic and $n_r(G/L) < n_r(G)$.*

An automorphism x of a group G is called a nil-automorphism if for every $g \in G$ there exists an integer $n = n(g)$ such that $[g,_n x] = 1$, where the commutator is taken in the holomorph of G. We will need the following result, which is an immediate consequence of [4, Theorem A].

PROPOSITION 2.15. *Any group of nil-automorphisms of a finite group is nilpotent.*

We are now in a position to complete the proof of Theorem A.

Proof of Theorem A. We will prove a formally stronger statement that the exponent of $w(G)$ is (e, w)-bounded for any $G \in Y(e, w)$. This will be sufficient for our purposes as we know that $X(e, w) \subseteq Y(e, w)$. Throughout, we use the fact that any quotient of a group in $Y(e, w)$ belongs to $Y(e, w)$.

Let r be as in Lemma 2.13. As r and $n_r(G)$ are (e, w)-bounded, we can use induction on $n_r(G)$. If $n_r(G) = (0, 0, 0)$, then G has odd order and so by the Feit-Thompson theorem [5] G is soluble. In this case the result holds by Proposition 2.11. So we may assume that $n_r(G) > (0, 0, 0)$.

First suppose that G has no nontrivial normal soluble subgroups. By Proposition 2.14 there exists a normal subgroup L in G such that L is residually monolithic and $n_r(G/L) < n_r(G)$. By induction the exponent of $w(G)L/L$ is (e, w)-bounded. By Lemma 2.12, applied with $u = w^e$, the exponent of L is also (e, w)-bounded so the result follows.

Now let us drop the assumption that G has no nontrivial normal soluble subgroups and let S be the soluble radical of G. Proposition 2.11 shows that the exponent of $w(S)$ is bounded. Thus, we pass to the quotient $G/w(S)$ and without loss of generality assume that $w(S) = 1$. As w has the property (bb), S has a characteristic subgroup N of bounded exponent such that S/N is nilpotent of bounded class. Passing to the quotient G/N we may assume that S is nilpotent of bounded class. Therefore $w_G(S)$ is a nilpotent subgroup of G generated by w-values. By Lemma 2.4 $w_G(S)$ has (e, w)-bounded exponent. Passing to the quotient $G/w_G(S)$ we may assume that $w_G(S) = 1$.

Suppose that w is the nth Engel word and let $a \in G$. Then $[S,_n a] \leq w_G(S) = 1$. Thus, every element of G induces a nil-automorphism of S and so by Proposition 2.15 $G/C_G(S)$ is nilpotent. Let $H = SC_G(S)$. As G/S has no nontrivial normal soluble subgroups, the same holds for H/S. We have $n_r(H/S) \leq n_r(H) \leq n_r(G)$. Since we know that the theorem holds for groups without nontrivial normal soluble subgroups, we conclude that $w(H)S/S$ is of bounded exponent. As S is nilpotent of bounded class we deduce that $S \leq Z_c(H)$ for some bounded integer c, whence $w(H)/Z_c(w(H))$ has bounded exponent. Lemma 2.1 applied to the group $w(H)$ tells us that

$\gamma_{c+1}(w(H))$ has bounded exponent. We pass to the quotient $G/\gamma_{c+1}(w(H))$ and without loss of generality assume that $\gamma_{c+1}(w(H)) = 1$. Now $w(H)$ is a nilpotent subgroup of G generated by w-values. By Lemma 2.4 $w(H)$ has (e,w)-bounded exponent and we may assume that $w(H) = 1$. In this case H is an n-Engel group. By Zorn's theorem [21] H is nilpotent. Moreover G/H is also nilpotent because $C_G(S) \leq H$ and so G is metanilpotent. Now the result follows from Proposition 2.11.

From now on we assume that $w = [x^n, y_1, y_2, \ldots, y_k]$. Since G/S has no nontrivial soluble normal subgroups and $n_r(G/S) \leq n_r(G)$, it follows that $w(G)S/S$ has bounded exponent. Put $K = G^n$. Recall that S contains no nontrivial w-values. Therefore $[K, S,_{k-1} G] = 1$ and so $S \cap K \leq Z_k(K)$. Combining the fact that $w(K)S/S$ has bounded exponent with Lemma 2.1 we deduce that $\gamma_{k+1}(w(K))$ has bounded exponent. Passing to the quotient over $\gamma_{k+1}(w(K))$ we may assume that $\gamma_{k+1}(w(K)) = 1$. This means that $w(K)$ is nilpotent, in which case it has bounded exponent since $G \in Y(e,w) \subseteq X(f,w)$ for a bounded integer f. Again, we may assume that $w(K) = 1$. Since $w(G) \leq K$, we have $w(w(G)) = 1$ and now the conclusion is immediate from Lemma 2.3. $\qquad\square$

3. Theorem B

LEMMA 3.1. *Let $G \in X(k,e,w)$. If N is a normal subgroup of G which contains no nontrivial w_e-values of G, then $[N, w(G)] = 1$.*

Proof. Let $x \in N$ and let y be a w-value. As $G \in X(k,e,w)$, the order of y divides e. Write $[y, x] = y^{-1}y^x = y^{e-1}y^x$. We see that $[y, x]$ is a w_e-value since y^{e-1} is a w_{e-1}-value and y^x is a w-value. By the hypothesis N contains no nontrivial w_e-values so $[y, x] = 1$ and the result follows. $\qquad\square$

The following proposition is immediate from Theorem 2 in [13].

PROPOSITION 3.2. *There exists a function $s(d)$ such that if G is a finite soluble group generated by $g_1, \ldots, g_d$, then every element of the derived group is equal to a product of $s(d)$ commutators of the form $[x, g]$, where $x \in G$ and $g \in \{g_1, \ldots, g_d\}$.*

The next proposition, whose proof is based on techniques developed in [6], is taken from [17].

PROPOSITION 3.3. *Let G be a finite group and $x \in G$. Suppose that every 4 conjugates of x generate a soluble subgroup of G of Fitting height at most h. Then $x \in F_h(G)$.*

LEMMA 3.4. *Given an integer e, there exist two integers h_0 and k_0 such that if w is a word and H is a normal subgroup of a group $G \in X(k,e,w)$ with $k \geq k_0$ such that $w_{e,G}(H)$ is soluble, then the Fitting height of $w_{e,G}(H)$ is at most h_0.*

Proof. Set $k_0 = s(4)e^2$ and choose $G \in X(k, e, w)$ with $k \geq k_0$. Suppose that H is a normal subgroup of G such that $w_{e,G}(H)$ is soluble and $g \in H$ is a w_e-value. Consider four arbitrary conjugates g_1, g_2, g_3, g_4 of g and let $T = \langle g_1, g_2, g_3, g_4 \rangle$. As $G \in X(k, e, w)$ the orders of g_i divide e. For every $x \in G$ we have $[x, g_i] = g_i^{-x} g_i = g_i^{(e-1)x} g_i$. It is clear that this is a w_{e^2}-value. By Proposition 3.2 every element of T' is the product of $s(4)$ commutators of the form $[x, g_i]$ for some $i = 1, \ldots, 4$. Therefore every element of T' is a $w_{e^2 s(4)}$-value. As $k \geq s(4)e^2$ and $G \in X(k, e, w)$ it follows that the derived subgroup T' of T has exponent dividing e. On the other hand, T/T' is abelian and generated by elements of order dividing e. Hence T has exponent dividing e^2. According to the Hall-Higman theory [7] it follows that T has Fitting height bounded by a constant h_0 (which depends on e only). Thus by Proposition 3.3 $g \in F_{h_0}(H)$. As this holds for every w_e-value g contained in H, it follows that $w_{e,G}(H) \leq F_{h_0}(H)$, so the Fitting height of $w_{e,G}(H)$ is at most h_0. $\qquad\square$

LEMMA 3.5. *Assume that $G \in X(k, e, w)$ and let t be an integer such that $1 \leq t \leq k/e$. If N is a normal subgroup of G, we have $G/N \in X(t, e, w)$.*

Proof. We put $\bar{G} = G/N$ and use the bar notation in the quotient. Let $\bar{H}$ be a nilpotent subgroup of $\bar{G}$ generated by w_t-values $\bar{a}_1, \ldots, \bar{a}_s$. As $\bar{H}$ is nilpotent and each $\bar{a}_i$ has order dividing e, a p-Sylow subgroup of $\bar{H}$ is of the form $\bar{P} = \langle \bar{a}_1^m, \ldots, \bar{a}_s^m \rangle$, where p is a prime dividing e and $e = p^\alpha m$, with $(m, p) = 1$. Let P be a p-Sylow subgroup of H (so that $PN/N = \bar{P}$) and let X be the set of the values of the word w_t^m in G. Then X is a normal subset of H consisting of p-elements. By [1, Lemma 2.1] $\bar{P} = \langle \overline{P \cap X} \rangle$. As $tm \leq te \leq k$, the subgroup $\langle P \cap X \rangle$ is a nilpotent subgroup of G generated by w_k values. Therefore $\langle P \cap X \rangle$ has exponent dividing e, whence also $\bar{P}$ has exponent dividing e. Thus, $\bar{H}$ has exponent dividing e, as required. $\quad\square$

In the next lemma we refine arguments used in Lemma 3.5 and Proposition 3.8 of [3].

LEMMA 3.6. *Assume that $G \in X(k, e, w)$ is a group without nontrivial normal soluble subgroups, where $k \geq e^3$. Suppose that s is a positive integer such that $k \geq es$. Then G possesses a normal subgroup L with the property that L is residually monolithic and G/L is a subdirect product of groups $G_i \in X(s, e/p_i, w)$, where p_i is a suitable prime divisor of e.*

Proof. Choose a minimal normal subgroup M in G. Write $M = S_1 \times \cdots \times S_t$, where the S_i's are isomorphic nonabelian simple subgroups and let L_M be the kernel of the natural permutation action of G on the set $\{S_1, \ldots, S_t\}$. We put $\bar{G} = G/L_M$ and use the bar notation in the quotient. Suppose first that $w_{e,G}(M) \neq 1$ Let b be a nontrivial w_e-value lying in M and write $b = b_1 \ldots b_t$, where $b_i \in S_i$. Choose j such that that $b_j \neq 1$. Since G acts on the set $\{S_1, \ldots, S_t\}$ transitively, without loss of generality we can assume that $j = 1$. Let now $y \in S_1$ be an element which does not centralize b_1.

Then $z = [b, y] = b^{-1}b^y = b_1^{-1}b_1^y$ is a non-trivial w_{e^2}-value lying in S_1. Since $k \geq e^3$ and $G \in X(k, e, w)$, the order of z divides e.

Let p be a prime dividing the order of z. Let $\bar{H}$ be a nilpotent subgroup of $\bar{G}$ generated by w_s-values $\bar{a}_1, \ldots, \bar{a}_v$. The subgroup $\bar{H}$ is the direct product of its Sylow subgroups. Let $\bar{R}$ be a r-Sylow subgroup of $\bar{H}$. If $e = r^\alpha m$ with $(r, m) = 1$, then $\bar{R} = \langle \bar{a}_1^m, \ldots, \bar{a}_v^m \rangle$. Denote by X the set of all values of the word w_s^m in G. As $s \leq k$, it follows that X is a normal subset of G consisting of r-elements of order dividing e.

Let R be an r-subgroup of G such that $RL_M/L_M = \bar{R}$ and let T be an r-Sylow subgroup of G containing R. Then $\langle T \cap X \rangle$ is a nilpotent subgroup of G generated by w_k-values and so it has exponent dividing e because $G \in X(k, e, w)$. By Lemma 2.1 of [1] $\langle \bar{T} \cap \bar{X} \rangle = \overline{\langle T \cap X \rangle}$. So $\bar{R} = \langle \bar{a}_1^m, \ldots, \bar{a}_v^m \rangle \leq \overline{\langle T \cap X \rangle}$ has exponent dividing e. If $r \neq p$, then the maximum power of r dividing e is the same as the maximum power of r dividing e/p, so $\bar{R}$ has exponent dividing e/p. Suppose now that $r = p$ and assume by contradiction that there exists $\bar{x} \in \bar{R}$ such that $\bar{x}^{e/p} \neq 1$. Recall that $e = p^\alpha m$. Hence $\bar{x}$ has order p^α. Since $\bar{x} \in \bar{R} \leq \overline{\langle T \cap X \rangle}$, we can choose $x \in \langle T \cap X \rangle$ such that $xL_M = \bar{x}$. The element x permutes regularly p^α subgroups among the S_i's and we can assume that S_1 is one of those. As $e^2 \leq k$, the element z is a w_k-value and therefore it has order dividing e. Set $z_1 = z^m$ and notice that z_1 is a nontrivial p-element which is also a w_k-value, because $e^2 m < e^3 \leq k$. It is clear that $T \cap M$ is a p-Sylow subgroup of M, so it is the direct product of suitable p-Sylow subgroups of S_i, one for each i. Thus we can conjugate z by a suitable element of S_1 and assume that $z_1 \in T$. As z_1 is a $w_{e^2}^m$-value, we see that $z_1 \in T \cap X$. Therefore $z_1 x \in \langle T \cap X \rangle$, which is a nilpotent subgroup of G generated by w_k-values. Hence, $z_1 x$ has order dividing e. In particular, as $z_1 x$ is a p-element, its order divides p^α. But

$$(z_1 x)^{p^\alpha} = z_1 z_1^{x^{-1}} z_1^{x^{-2}} \ldots z_1^x$$

and as all different conjugates of z_1 in the above expression lie in different factors S_i's, it follows that $(z_1 x)^{p^\alpha} \neq 1$, a contradiction. Thus, $\bar{G} \in X(s, e/p, w)$.

Now consider the case where our minimal normal subgroup M has the property that $w_{e,G}(M) = 1$. By Lemma 3.1 we have $[w(G), M] = 1$ and therefore $w(G) \leq L_M$. In this case $\bar{G} \in X(e^2, 1, w) \leq X(s, e/p_i, w)$ for any prime divisor p_i of e.

Let L be the intersection of all the subgroups L_M where M ranges through the minimal normal subgroups of G. Then G/L is a subdirect product of the groups G/L_M, so we just need to show that L is residually monolithic. If N is the product of all minimal normal subgroups of G, it is clear that N is the direct product of pairwise commuting simple groups $N_1, \ldots, N_l$ and L is the intersection of the normalizers of the N_i's. The subgroup L acts on each N_i by conjugation and let $\rho_i : N \to \mathrm{Aut}(N_i)$ be the natural homomorphism. It is easy to see that the image of ρ_i is monolithic. Since G has no nontrivial normal soluble subgroups it follows that $C_G(N) = 1$ and

so L embeds into the direct product of the groups $\rho_i(N)$ for $i = 1, \ldots, l$. The lemma follows. $\square$

Now we are ready to present a proof of Theorem B.

Proof of Theorem B. We will use induction on e to prove that there exist two numbers $k = k(e)$ and $E = E(e, w)$ such that if $G \in X(k, e, w)$, then $w(G)$ has exponent dividing E. If $e = 1$, the result is trivial. Therefore we assume that $e \geq 2$ and there exist constants k_1 and E_1 such that the exponent of $w(D)$ divides E_1 whenever $D \in X(k_1, e/p, w)$ for some prime divisor p of e. We let k_0 and h_0 have the same meaning as in Lemma 3.4 and denote by m the maximum of $\{e^2, k_0, k_1\}$. By Lemma 2.12 there exists a constant E_0 depending only on w and e such that the exponent of any residually monolithic group satisfying the law $w^e \equiv 1$ has exponent dividing E_0.

We will show first that the constants $k_2 = me$ and $E_2 = E_0 E_1$ have the property that if G is a finite group in $X(k_2, e, w)$ without nontrivial normal soluble subgroups, then the exponent of the verbal subgroup $w(G)$ divides E_2. Thus, take $G \in X(k_2, e, w)$ and assume that G has no nontrivial normal soluble subgroups. By Lemma 3.6 applied with $s = m$, G possesses a normal subgroup L such that L is residually monolithic and G/L is the subdirect product of groups $G_i \in X(m, e/p_i, w)$, where p_i is a suitable prime dividing e. By induction the exponent of $w(G_i)$ divides E_1 for every i and we know that the exponent of L divides E_0. It follows that indeed the exponent of $w(G)$ divides E_2.

We will now deal with the group $G \in X(k_2, e, w)$ but we drop the assumption that G has no nontrivial normal soluble subgroups. The symbol $S(D)$ will stand for the soluble radical of a group D. Let $S = S(G)$ and $T = w_{e,G}(S)$. Since $k_2 \geq k_0$, by Lemma 3.4 it follows that the Fitting height $h = h(T)$ of T is at most h_0. From now on we will be using h as a second induction parameter.

Consider the case where $h = 0$. This happens if and only if $T = 1$. By Lemma 3.1 $[S, w(G)] = 1$. Therefore every nilpotent subgroup generated by w_{k_2}-values in G/S is an image of a nilpotent subgroup generated by w_{k_2}-values in G. Hence it must have exponent dividing e. It follows that G/S belongs to $X(k_2, e, w)$. Since G/S has no nontrivial normal soluble subgroups, the above argument shows that $w(G/S)$ has exponent dividing E_2. Since $[S, w(G)] = 1$, it follows that $w(G)/Z(w(G))$ has exponent dividing E_2. By Lemma 2.1 we conclude that $w(G)'$ has (e, w)-bounded exponent. As $w(G)/w(G)'$ is abelian and generated by elements of order dividing e, it has exponent dividing e. Therefore the exponent of $w(G)$ is (e, w)-bounded. Thus, in the case where $h = 0$ the result is established.

We will now assume that $h \geq 1$. The induction hypothesis will be that there exist two numbers $k_3 \geq k_0$ and E_3 such that if $D \in X(k_3, e, w)$ and $h(w_{e,D}(S(D))) \leq h - 1$, then $w(D)$ has exponent dividing E_3. Put $k_4 = ek_3$ and assume that our group G belongs to $X(k_4, e, w)$.

Let F be the Fitting subgroup of T and $M = w_{e,G}(F)$. Since M is nilpotent, it has exponent dividing e. So we can consider $\bar{G} = G/M$ and it is enough to prove that $w(\bar{G})$ has bounded exponent. By Lemma 3.1, $[\bar{F}, w(\bar{G})] = 1$. This implies in particular that $[\bar{F}, \bar{T}] = 1$. Hence, $\bar{F} \leq Z(\bar{T})$ and so the Fitting height of $\bar{T}$ is smaller than h. Moreover, by Lemma 3.5, $\bar{G} \in X(k_3, e, w)$ and so by induction the exponent of $w(\bar{G})$ divides E_3. Therefore the exponent of $w(G)$ divides eE_3 and this completes the induction step. As $h \leq h_0$ is e-bounded, the theorem follows. $\qquad\square$

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